D Dirac operator

D^2 Laplacian

McKean-Singer : X cpt of dim m = even, for $t>0$

$$\mathrm{Ind}(D_+) = \mathrm{Tr}_s[\exp(-tD^2)] = \int_X \mathrm{Tr}_s^E[P_t(x,x)]\,dv(x)$$

Thm (heat kernel expansion)

$\exists\ a_j \in C^\infty(X, \mathrm{End}(E))$ s.t. $\forall N$

$$(*) \qquad P_t(x,x) = t^{-\frac{m}{2}} \sum_{j=0}^{N} a_j(x) t^j + O(t^{N+1-\frac{m}{2}})$$

uniformly for $t \in (0, \frac{1}{2}]$, $x \in X$

Thm (local index theorem)

$$\mathrm{Tr}_s^E[a_j(x)]\,dv(x) = \begin{cases} 0 & j < \frac{m}{2} \\ [\hat{A}(TX, \nabla^{TX})\, ch^{E/S}(E, \nabla^E)]^{max} & j = \frac{m}{2} \end{cases}$$

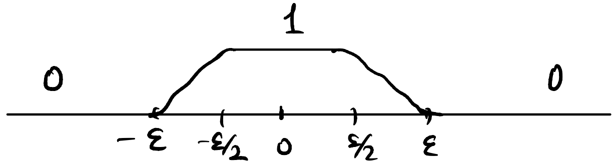
Proof Step 1 : Localization of heat kernel $P_t(x,x)$ by the functional calculus of D^2

Fix $\varepsilon > 0$ small

$f: \mathbb{R} \to [0,1]$ C^∞ even

s.t. $f(v) = \begin{cases} 1 & |v| \leq \varepsilon/2 \\ 0 & |v| \geq \varepsilon \end{cases}$ $\qquad f(v) = f(-v)$

f



Def: For $u>0$, $a\in\mathbb{R}$

$$F_u(a) := \int_{-\infty}^{+\infty} \cos(sa)\, e^{-s^2/2} f(\sqrt{u}\,s)\frac{ds}{\sqrt{2\pi}}$$

$$G_u(a) := \int_{-\infty}^{+\infty} \cos(sa)\, e^{-s^2/2} \left(1-f(\sqrt{u}\,s)\right)\frac{ds}{\sqrt{2\pi}}$$

① $F_u(a)$, $G_u(a)$ are C^∞ even function on $a\in\mathbb{R}$ non-negative.

② $$F_u(a)+G_u(a) = \int_{\mathbb{R}} \cos(sa)\, e^{-s^2/2}\frac{ds}{\sqrt{2\pi}}$$

$$= \int_{\mathbb{R}} e^{\sqrt{-1}sa - s^2/2}\frac{ds}{\sqrt{2\pi}}$$

$$= \int_{\mathbb{R}} e^{-\frac{1}{2}(s-\sqrt{-1}a)^2 - \frac{a^2}{2}}\frac{ds}{\sqrt{2\pi}}$$

$$= e^{-a^2/2}$$

Now we can define the operators acting on $L^2(X,E)$

$F_u(tD)$ and $G_u(tD)$ s.t.

$$\begin{cases} F_u(tD)\big|_{H_\lambda} = F_u(t\sqrt{\lambda})\,Id_{H_\lambda} \\ G_u(tD)\big|_{H_\lambda} = G_u(t\sqrt{\lambda})\,Id_{H_\lambda} \end{cases}$$

$H_\lambda = \lambda$-eigenspace of D^2

Then $\underset{\sqrt{u}D}{F_u(tD)} + \underset{\sqrt{u}D}{G_u(tD)} = \bigoplus_\lambda \left(F_u(t\sqrt{\lambda}) + G_u(t\sqrt{\lambda})\right) Id_{H_\lambda}$

$$= \bigoplus_\lambda e^{-\frac{(t\sqrt{\lambda})^2}{2}} Id_{H_\lambda}$$

$$e^{-\frac{(t\sqrt{\lambda})^2}{2}} = e^{-t\lambda/2}$$

$$= \exp(-tD^2/2). \qquad \exp(-uD^2/2)$$

Schwartz kernels $F_u(tD)(x,x')$, $G_u(tD)(x,x')$

C^∞ on $X \times X$

e.g. $F_u(tD)(x,x') = \sum_j F_u(t\sqrt{\lambda_j})\,\phi_j(x) \otimes \phi_j(x')^*$

Prop: $\exists\, C_1, C_2 > 0$ s.t. $\forall\, u \in (0,1]$, $x, x' \in X$,

$$\left|G_u(\sqrt{u}D)(x,x')\right| \le C_1 e^{-C_2/u}$$

Pf: For any $a \in \mathbb{R}$ $\qquad v = \sqrt{u}s$

$$G_u(\sqrt{u}a) = \int_{\mathbb{R}} \cos(s\sqrt{u}a)\, e^{-s^2/2}\,(1 - \underbrace{f(\sqrt{u}s)}_{=1 \text{ if } |\sqrt{u}s| \le \varepsilon/2})\,\frac{ds}{\sqrt{2\pi}}$$

$$= \int_{|v| \ge \varepsilon/2} \cos(va)\, e^{-v^2/2u}\,(1 - f(v))\,\frac{dv}{\sqrt{2\pi u}}$$

For any $k \in \mathbb{N}$

$$a^{2k} G_u(\sqrt{u}\,a) = \int_{|\nu| \geq \varepsilon/2} \underbrace{a^{2k} \cos(\nu a)}_{\substack{\| \\ (-1)^k \left(\frac{\partial}{\partial \nu}\right)^{2k} \cos(\nu a)}} e^{-\frac{\nu^2}{2u}} (1 - f(\nu)) \frac{d\nu}{\sqrt{2\pi u}}$$

by the fact that

$\left(\frac{\partial}{\partial \nu}\right)^{\ell} (1 - f(\nu)) \Big|_{|\nu| = \frac{\varepsilon}{2}} \equiv 0$

$\ell \in \mathbb{N}_{\geq 1}$

$$= \int_{|\nu| \geq \varepsilon/2} \cos(\nu a) \left(\frac{\partial}{\partial \nu}\right)^{2k} \left(e^{-\frac{\nu^2}{2u}} (1 - f(\nu)) \right) \frac{d\nu}{\sqrt{2\pi u}}$$

$$= \int_{|\nu| \geq \varepsilon/2} \cos(\nu a)\, e^{-\frac{\nu^2}{2u}} \sum_{\ell=0}^{2k} \left(\frac{\partial}{\partial \nu}\right)^{\ell} (1 - f(\nu)) \underbrace{P_\ell\left(\frac{1}{u}, \nu\right)}_{\text{polynomial in } \frac{1}{u} \,\&\, \nu} \frac{d\nu}{\sqrt{2\pi u}}$$

$\sup_{0 < u \leq 1} \frac{1}{u^{\ell}} e^{-\frac{\varepsilon^2}{16u}} < +\infty$

$$\leq C_k\, e^{-\frac{\varepsilon^2}{16u}}$$

for some $C_k > 0$.

This implies ~~that~~, $\forall k$, $\exists\, C > 0$ s.t.

$$\left\| D^{2k} G_u(\sqrt{u} D) \right\|^{0,0}_{L^2} \leq C_k\, e^{-\frac{\varepsilon^2}{16u}}$$

operator norm $L^2 \to L^2$

$\| D^{2k} \cdot \|_{L^2} \Longleftrightarrow W^{2k}$ Sobolev norm

Sobolev embedding $W^{2k} \hookrightarrow C^{2k - \frac{m}{2}}(X, E)$

$$\Rightarrow \quad \left| G_u(\sqrt{u} D)(x, x') \right| \leq c_1 e^{-c_2/u} \quad \text{for } k \gg 1$$

#

5

Prop : (Finite propagation speed)

Note ~~that~~ $\operatorname{supp} f \subset [-\varepsilon, \varepsilon]$, given any $x \in X$

Then $\operatorname{supp} F_u(\sqrt{u}D)(x, \cdot) \subset \bar{B}^X(x, \varepsilon)$

$= \{ y \in X : dist(x, y) \le \varepsilon \}$

Moreover $F_u(\sqrt{u}D)(x, \cdot)$ depends only on $D^2|_{\bar{B}^X(x, \varepsilon)}$.

Reason : wave operator, for $x_0 \in X$

$$\omega_t(y) = \cos(t|D|)(x_0, y)$$

s.t.

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + D_y^2\right) \omega_t(y) = 0 \\ \lim_{t \to 0} \omega_t(y) = \delta_{x_0}(y), \quad \lim_{t \to 0} \frac{\partial}{\partial t} \omega_t(y) = 0 \end{cases}$$

Finite propagation speed

$$\Rightarrow \begin{cases} \operatorname{supp} \omega_t(y) \subset \bar{B}^X(x_0, t) \\ \text{it depends only on } D^2|_{\bar{B}^X(x_0, t)} \end{cases}$$

$$F_u(\sqrt{u}D)(x_0, y) = \int_{\mathbb{R}} \underbrace{\cos(s\sqrt{u}D)(x_0, y)}_{\subset\, \bar{B}^X(x_0, \sqrt{u}s)} e^{-s^2/2} \underbrace{f(\sqrt{u}s)}_{=\,0 \text{ for } |\sqrt{u}s| \ge \varepsilon} \frac{ds}{\sqrt{2\pi}}$$

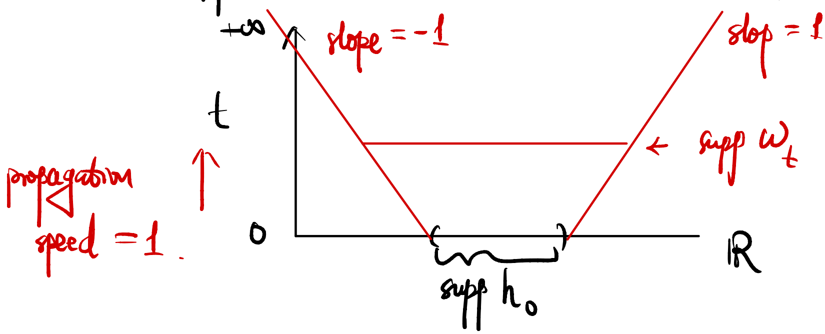
Ref to Energy estimates for wave equation

On $\mathbb{R}_+ \times \mathbb{R}$ (t, x)

$$\begin{cases} \left(\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2\right) \omega_t(x) = 0 \\ \omega_0(x) = h_0(x) \\ \frac{\partial}{\partial t}\Big|_{t=0} \omega_t(x) = 0 \end{cases}$$

$$\Rightarrow \quad w_t(x) = \frac{1}{2}\left(h_0(x+t) + h_0(x-t) \right)$$

$$\operatorname{supp} w_t(x) = \{ y \in \mathbb{R} : \operatorname{dist}(y, \operatorname{supp} h_0) \le t \}$$



As a consequence

$$e^{-uD^2/2}(x,x) = F_u(\sqrt{u}D)(x,x) + O(e^{-c/u})$$

As $u \to +\infty$, $\quad F_u(\sqrt{u}D)(x,x)$ depends only on $D^2|_{B^X(x,\varepsilon)}$

<u>Proof step 2</u>: Localization of D^2

$(X, g^{TX}) \rightsquigarrow$ Levi-Civita connection ∇^{TX}

For any $x \in X$, $v \in T_xX$, $\exists !$ geodesic $\gamma : (-\delta, \delta) \to X$ s.t.

$\gamma(0) = x$ $\qquad \nabla^{TX}_{\dot\gamma}\dot\gamma = 0$

$\gamma'(0) = v$

Then we can define the exponential map

$$T_xX \supset B^{T_xX}(0,\delta) \longrightarrow B^X(x,\delta) \subset X$$

$$v \longmapsto \exp_x(v)$$

where $\exp_x(v) = \gamma(1)$

for the geodesic γ with $\begin{cases} \gamma(0) = x \\ \gamma'(0) = v \end{cases}$

Prop: when $\delta > 0$ is sufficiently small, $\forall x \in X$

$$\exp_x : B^{T_xX}(0, \delta) \xrightarrow{\sim} B^X(x, \delta)$$

defines a local diffeomorphism

This is call geodesic coordinates centered at x.

We fix $\varepsilon > 0$ s.t. $4\varepsilon < \delta$.

we also fix $x_0 \in X$. Consider the local chart

$$B^{T_{x_0}X}(0, 4\varepsilon) \longrightarrow B(x_0, 4\varepsilon)$$

$$\begin{matrix} \cup \\ Z \end{matrix}$$

Locally $E = S^{TX} \hat{\otimes} W$

$\nabla^E \quad \nabla^{S^{TX}} \otimes 1 + 1 \otimes \nabla^W$

We trivialize $E|_{B(x_0, 4\varepsilon)}$ as $B(x_0, 4\varepsilon) \times E_{x_0}$

$S^{TX} \qquad S^{TX}_{x_0}$

$W \qquad W_{x_0}$

via $\forall Z \in B^{T_{x_0}X}(0, 4\varepsilon)$

parallel transport along the path

$$[0,1] \ni s \mapsto sZ$$

w.r.t. ∇^E, $\nabla^{S^{TX}}$, ∇^W.

this also identifies the Hermitian metrics.

Fix $\{e_j\}$ ONB of $(T_{x_0}X, g^{T_{x_0}X})$

e.g. $e_j = \frac{\partial}{\partial Z_j}$

$\{\tilde{e}_j\}$ orthonormal frame of $(TX, g^{TX})\big|_{B(x_0, 4\varepsilon)}$

obtained by parallel transport of e_j along

$[0,1] \ni s \mapsto sZ$

Lichnerowicz formula :

$$D_Z^2 = \Delta_Z^E + \frac{r^X(Z)}{4} + \frac{1}{2}\sum_{j,k} R_Z^W(\tilde{e}_j, \tilde{e}_k)\, c(\tilde{e}_j)\, c(\tilde{e}_k)$$

Note that $e_j = \frac{\partial}{\partial Z_j}$ local frame of TX on $B(x, 4\varepsilon)$

NOT always orthonormal

$$g_{jk} = g_Z^{TX}(e_j, e_k)$$

$$\Delta^E = -\sum_{j,k} g^{jk}\left(\nabla_{e_j}^E \nabla_{e_k}^E - \nabla^E_{\nabla^{TX}_{e_j} e_k}\right)$$

With respect to the local trivialization of E, S^{TX}, W

write
$$\begin{cases} \nabla^E = d + \Gamma^E \\ \nabla^{S^{TX}} = d + \Gamma^{S^{TX}} \\ \nabla^W = d + \Gamma^W \end{cases}$$

s.t. $\quad \Gamma^{\bullet}_{Z=0} = 0$

Take $X_0 = T_{x_0}X \cong \mathbb{R}^m$

Fix g^{TX_0} s.t. $\begin{cases} g^{TX_0}|_{B^{X_0}(0,2\varepsilon)} = g^{TX} \\ g^{TX_0}|_{B^{X_0}(0,4\varepsilon)^c} = g^{TX}_{x_0} \end{cases}$

Riemannian metric on X_0

Def: $dv_{X_0}(Z) = \kappa(Z)\, dv_{T_{x_0}X}(Z)$

$$\begin{array}{ccc} dv_{X_0} & & dv_{T_{x_0}X} \\ \updownarrow & & \updownarrow \\ g^{TX_0} & & g^{TX}_{x_0} \end{array}$$

$\kappa \in C^\infty(X_0, \mathbb{R}_{>0})$ with $\kappa(0) = 1$.

Now $\rho \in C^\infty(X_0, [0,1])$ s.t.

$$\rho|_{B^{X_0}(0,2\varepsilon)} \equiv 1, \quad \rho|_{B^{X_0}(0,4\varepsilon)^c} \equiv 0$$

For Clifford bundle $E_0 := S^{TX}_{x_0} \hat{\otimes} W_{x_0}$ on X_0

$$\nabla^{E_0} := d + \rho(Z)(\underbrace{\Gamma^{S^{TX}} + \Gamma^W}_{\Gamma^E})$$

Hermitian connection on $(E_0 := E_{x_0},\ h^{E_0} := h^E_{x_0})$

Def: $L := \Delta^{E_0} + \rho(Z)\left(\frac{r^X}{4} + \frac{1}{2}\sum_{j,k} R^W(\tilde{e}_j, \tilde{e}_k)\, c(\tilde{e}_j)\, c(\tilde{e}_k)\right)$

acting on $C^\infty(X_0, E_0)$

Δ^{E_0} Bochner Laplacian ass. with ∇^{E_0}

Prop: $\begin{cases} L|_{B^{X_0}(0,2\varepsilon) \simeq B^X(x_0,2\varepsilon)} = D^2|_{B^X(x_0,2\varepsilon)} \\ L|_{B^{X_0}(0,4\varepsilon)^c} = -\sum_j \left(\frac{\partial}{\partial Z_j}\right)^2 \end{cases}$